Master's Thesis

Robust Constraint Satisfaction for C/GMRES based on NMPC with Parameter Uncertainties

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Abstract

Using optimization for solving control problems has become much more accessible due to computational advancements in recent years. With the introduction of MPC in the petrochemical industry for linear plant models, nonlinear applications followed and enabled finding optimal control solutions for complex tasks. These tasks often require the abidance by certain limitations, which are referred to as constraints in an NMPC environment.

This development entailed two major hurdles to be negotiated. The first one is finding efficient algorithms for problems such as NMPC optimizations, where Ohtsuka's C/GMRES provides a remedy and is heavily used in this work. Second, improving model accuracy is an ongoing topic, which tries to eliminate model-plant mismatches, but usually fails to do so, because models solely approximate real plants. Hence, robust control approaches were introduced in order to deal with disturbances and inaccuracies.

In this work, ideas from robust control are used and developed in order to deal with the problem of robustly satisfying inequality constraints on models with parameter uncertainties. Three approaches are presented, where the main goal is to find a worst case from a given parameter set. One method makes use of the fact that quasi-convex functions find their maximum at an extreme point of the function's argument and therefore enables finding the worst case via extreme point scenarios. Next, a method to find the worst case directly from the continuous parameter set is discussed, which is treated as a separate maximization problem, resulting in a bilevel optimization problem. A way of transforming such problems into singlelevel tasks via KKT conditions is presented, requiring the constraint function to be pseudo-concave. At last, sensitivity is discussed and used as a separate tool and to facilitate previous approaches.

Introduction

Originally a 'robust control system' was understood to be maintaining stability and performance specifications for a certain range of model variations and noise signals. Ever since MPC was introduced, constraints attracted attention and inevitably the term robustness was widened to incorporate robust constraint fulfilment. Robust constraint fulfilment in general means satisfying constraints while model plant mismatches or other disturbances are present. Neglecting signal noise, the problem being tackled deals with model uncertainties — parameter uncertainties to be precise. This means that the plant model is known, bar some parameters which are only guaranteed to lie within a certain range.

Problem Formulation

By dividing the classical NMPC problem into two parts, where one keeps track of a trajectory based on nominal parameters $\tilde{\mathbf{p}}$, while the other enables calculating with an arbitrary $\mathbf{p}$, the continuous problem formulation can be deduced:

$$\begin{aligned}\min_{\mathbf{u}}\ & J(\tilde{\mathbf{x}},\mathbf{u}) = \varphi(\tilde{\mathbf{x}}_N) + \textstyle\sum_{i=0}^{N-1} L(\tilde{\mathbf{x}}_i,\mathbf{u}_i)\\ \text{s.t.}\ & \mathbf{x}_{i+1} = \mathbf{x}_i + \mathbf{f}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p}) \quad \forall\, i\in\mathcal{I} = \{0,1,\dots,N-1\}\\ & \tilde{\mathbf{x}}_{i+1} = \tilde{\mathbf{x}}_i + \mathbf{f}(\tilde{\mathbf{x}}_i,\mathbf{u}_i,\tilde{\mathbf{p}}) \quad \forall\, i\in\tilde{\mathcal{I}} = \{0,1,\dots,N-1\}\\ & \mathbf{g}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p}) \le \mathbf{0} \quad \forall\, i\in\mathcal{I}_{+1} = \{1,2,\dots,N\} \quad \forall\, \mathbf{p}\in\mathcal{P}_\infty,\end{aligned}$$

Here, $\mathbf{p}$ is chosen to be confined by $\mathbf{p}\in\mathcal{P}_\infty$, where $\mathcal{P}_\infty := \{\mathbf{p} : \mathbf{p}_{\text{min}} \le \mathbf{p} \le \mathbf{p}_{\text{max}}\}$. However in some cases, $\mathcal{P}_\infty$ might be discrete. This can be the case, when there is a known number of devices to be controlled with parameters that were measured beforehand. Therefore, we introduce a set $\mathcal{P} := \{\mathbf{p}^1,\mathbf{p}^2,\dots,\mathbf{p}^\nu\} = \mathbf{P}$ that contains every parameter vector $\mathbf{p}^k$ obtained.

$$\begin{aligned}\min_{\mathbf{u}}\ & J(\tilde{\mathbf{x}},\mathbf{u})\\ \text{s.t.}\ & \mathbf{X}_{i+1} = \mathbf{X}_i + \mathbf{F}(\mathbf{X}_i,\mathbf{u}_i,\mathbf{P}) \quad \forall\, i\in\mathcal{I} = \{0,1,\dots,N-1\}\\ & \tilde{\mathbf{x}}_{i+1} = \tilde{\mathbf{x}}_i + \mathbf{f}(\tilde{\mathbf{x}}_i,\mathbf{u}_i,\tilde{\mathbf{p}}) \quad \forall\, i\in\tilde{\mathcal{I}} = \{0,1,\dots,N-1\}\\ & \mathbf{G}(\mathbf{X}_i,\mathbf{u}_i,\mathbf{P}) \le \mathbf{0} \quad \forall\, i\in\mathcal{I}_{+1} = \{1,2,\dots,N\}.\end{aligned}$$

Since the plant is only affected by the optimization's first entry $\mathbf{u}_0$, the problem can be relaxed by choosing $\mathcal{I} := \{0\}$.

Scenario Based Approach

The goal is to find the discrete worst case parameter according to the following definition:

Definition 1. *The parameter (vector) $\mathbf{p}_\nu^\star$ is called the discrete worst case parameter (vector) of a constraint function $g_\nu(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p})$ iff*

$$\mathbf{p}_\nu^\star := \arg\max_{\mathbf{p}} g_\nu(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p}) \quad s.t.\ \mathbf{p}\in\mathcal{P}$$

Irrespective of any previous knowledge, if $\mathcal{P}_\infty := \{\mathbf{p} : \mathbf{p}_{\text{min}} \le \mathbf{p} \le \mathbf{p}_{\text{max}}\}$ holds, it seems reasonable to set $\mathcal{P} := \{\mathbf{p}_{\text{min}},\dots,\mathbf{p}_{\text{max}}\}$, i.e. containing the extreme points:

Theorem 1. *Let f be a quasi-convex function defined on the bounded, closed convex set Ω. If f has a maximum over Ω it is achieved at an extreme point of Ω.*

and therefore

Theorem 2. *If $g_\nu(\mathbf{x}_0,\mathbf{u}_i,\mathbf{p}) \le \mathbf{0}$ are quasi-convex inequality constraint functions in $\mathbf{p}\in\mathcal{P}_\infty \quad \forall\,\mathbf{x}_0\in\mathcal{X}^\star\subseteq\mathcal{X}, \mathbf{u}_i\in\mathcal{U}^\star\subseteq\mathcal{U}$, where $\mathcal{P}_\infty$ is a bounded, closed convex set, the worst case, which is the maximum of $g_\nu(\mathbf{x}_0,\mathbf{u}_i,\mathbf{p})$, can be established by only considering the extreme points of $\mathcal{P}_\infty$.*

Min-Max Based Approach

The goal is to find the continuous worst case parameter according to the following definition:

Definition 2. *The parameter (vector) $\mathbf{p}_\nu^\star$ is called the continuous worst case parameter (vector) of a constraint function $g_\nu(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p})$ iff*

$$\mathbf{p}_\nu^\star := \arg\max_{\mathbf{p}} g_\nu(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p}) \quad s.t.\ \mathbf{p}\in\mathcal{P}_\infty$$

First, the NMPC problem is altered in order to incorporate the definition of the continuous worst case:

$$\begin{aligned}\min_{\mathbf{u}}\ & J(\tilde{\mathbf{x}},\mathbf{u})\\ \text{s.t.}\ & \tilde{\mathbf{x}}_{i+1} = \tilde{\mathbf{x}}_i + \mathbf{f}(\mathbf{x}_i,\mathbf{u}_i,\tilde{\mathbf{p}}) \quad \forall\, i\in\tilde{\mathcal{I}}\\ & \boldsymbol{\delta}^\star \le \mathbf{0}\\ & (\mathbf{p}^\star,\delta_\nu^\star) := \arg\max_{\mathbf{p},\delta_\nu} \delta_\nu \quad \forall\,\nu\in\{1,\dots,r\}\\ & \qquad \text{s.t.}\ \mathbf{x}_{i+1} = \mathbf{x}_i + \mathbf{f}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p}) \quad \forall\, i\in\tilde{\mathcal{I}}\\ & \qquad \mathbf{g}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p}) = \boldsymbol{\delta}\\ & \qquad \mathbf{p}\in\mathcal{P}_\infty,\end{aligned}$$

which can be turned into a single-level problem by utilizing KKT conditions, requiring the constraint function to be pseudo-concave for a global maximum.

Sensitivity Based Approach

The goal of the sensitivity based approach is to minimize the parameter uncertainties' impact, which however does not guarantee robust constraint satisfaction, but might still be a sufficient approach for some applications. Therefore, we introduce the term constraint sensitivity derived from performance related ideas:

$$\mathbf{S}_\text{C} = \frac{\text{d}\mathbf{g}}{\text{d}\mathbf{p}} = \frac{\partial\mathbf{g}}{\partial\mathbf{p}} + \frac{\partial\mathbf{g}}{\partial\mathbf{x}_i}\frac{\partial\mathbf{x}_i}{\partial\mathbf{p}}$$

In order to minimize $\mathbf{S}_\text{C}$, a new performance index is defined:

$$\bar{L} = L + L_\text{S},$$

with

$$L_\text{S} = \mathbf{w}_\text{S}{}^T\mathbf{S}_\text{C}{}^T\mathbf{S}_\text{C}\mathbf{w}_\text{S}.$$

By introducing sensitivity states, one can define new state vectors to calculate $\mathbf{S}_\text{C}$:

$$\bar{\mathbf{x}}_i = \left[\mathbf{x}_i^T, \left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{p}}\right)^T\right]^T$$

$$\bar{\mathbf{f}}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p}) = \left[\mathbf{f}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{p})^T, \left(\frac{d}{dt}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{p}}\right)\right)^T\right]^T$$

Summary

The first method, called Scenario Based Approach, analysed how a worst case parameter can be extracted from or included in a discretized parameter set. A guideline on how to choose a parameter set for robustly satisfying constraints was given and the necessary requirements were analysed. It was shown that the developed method culminates in requiring checking the parameter's extreme points, which guarantees robust results as long as a quasi-convex constraint function is present.

The second method, named minmax, utilizes the same concept of trying to find a worst case parameter, but uses a nested optimization over a continuous parameter set instead of discretizing the set in the first place. An algorithm, which uses KKT conditions to solve the inner program, was developed. The choice of utilizing KKT conditions enforces a pseudo-concave property on the constraint functions to guarantee global maxima, setting the minmax approach apart from the scenario method.

The third method, labelled sensitivity approach, was introduced with the idea of minimizing the parameter uncertainties' impact on the constraint functions. The preliminary method is simple and solely requires altering the cost function to incorporate the sensitivity with a weighting factor.